OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4758/01**

Differential Equations

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Wednesday 18 May 2011
Morning

Duration: 1 hour 30 minutes

* 4 7 5 8 0 1 *

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72.**
- This document consists of **4** pages. Any blank pages are indicated.

1 The differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 13\cos 2t \qquad (*)$$

is to be solved.

(i) Find the general solution. **[9]**

(ii) Find the particular solution, given that when $t = 0$, y and $\frac{dy}{dt}$ are both zero. **[6]**

Now consider the differential equation

$$\frac{d^3z}{dt^3} + 4\frac{d^2z}{dt^2} + 3\frac{dz}{dt} = -26\sin 2t.$$

(iii) Show that the general solution may be expressed as $z = y + c$ where y is the general solution of $(*)$ and c is a constant. **[2]**

(iv) When $t = 0$, $z = 2$, $\frac{dz}{dt} = 0$ and $\frac{d^2z}{dt^2} = 13$. Use these conditions to find the particular solution. **[7]**

2 **(a)** A curve in the x-y plane satisfies the differential equation

$$\frac{dy}{dx} - \frac{2y}{x} = \sqrt{x}$$

for $x > 0$.

(i) Find the general solution for y in terms of x. **[8]**

The curve passes through (1, 0).

(ii) Find the equation of this curve. **[2]**

(iii) Find the coordinates of the stationary point of this curve and find the values to which y and $\frac{dy}{dx}$ tend as $x \to 0$. Sketch the curve. **[6]**

(b) The differential equation

$$\frac{dy}{dx} = \sqrt{x^2 + y^2}$$

is to be solved approximately by using a tangent field.

(i) Describe the shape of the isocline for which $\frac{dy}{dx} = 1$. **[2]**

(ii) Sketch, on the same axes, the isoclines for the cases $\frac{dy}{dx} = 1$, $\frac{dy}{dx} = 2$, $\frac{dy}{dx} = 3$. Use these isoclines to draw a tangent field. **[3]**

(iii) Sketch the solution curve through (0, 1). **[1]**

(iv) Sketch the solution curve through the origin. **[2]**

3 **(a)** A particle of mass 2 kg moves on a horizontal straight line containing the origin O. When its displacement is x m from O, it is subject to a force of magnitude $2k^2x$ N directed towards O, where k is a positive constant.

(i) Show that the velocity, v m s^{-1}, of the particle satisfies the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -k^2x.$$ **[3]**

The particle is at rest when $x = a$, where a is a positive constant.

(ii) Solve the differential equation, subject to this condition. Hence show that, while the particle moves in the negative direction,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k\sqrt{a^2 - x^2}.$$ **[6]**

Initially the particle is at $x = a$.

(iii) Use the standard integral

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c$$

to find x in terms of t, k and a. **[5]**

(b) At time t s, the angle, θ rad, that a pendulum makes with the vertical satisfies the differential equation

$$\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = -9\sin\theta$$

where $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$.

(i) Solve the differential equation for ω in terms of θ subject to the condition $\omega = 0$ when $\theta = \frac{1}{3}\pi$. Hence show that, while θ is decreasing,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -3\sqrt{2\cos\theta - 1}.$$ **[6]**

(ii) Starting from $\theta = \frac{1}{3}\pi$ when $t = 0$, use Euler's method with a step length of 0.1 to estimate θ when $t = 0.1$. The algorithm is given by $t_{r+1} = t_r + h$, $\theta_{r+1} = \theta_r + h\dot{\theta}_r$. State whether this algorithm can usefully be continued, justifying your answer. **[4]**

[Question 4 is printed overleaf.]

4 The quantities x and y at time t are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = -3x - 2y + 3t,$$
$$\frac{dy}{dt} = 2x + y + t + 2.$$

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = -5t - 1$. **[5]**

(ii) Find the general solution for x. **[8]**

(iii) Find the corresponding general solution for y. **[4]**

When $t = 0$, $x = 9$ and $y = 0$.

(iv) Find the particular solutions. **[4]**

(v) Find approximate expressions for x and y in terms of t, valid for large positive values of t. **[3]**

RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4758:** Differential Equations

Mark Scheme for June 2011

Oxford Cambridge and RSA Examinations

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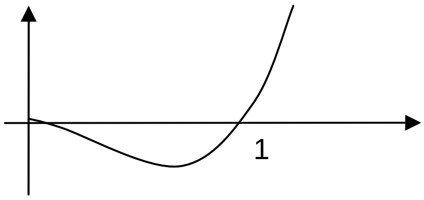
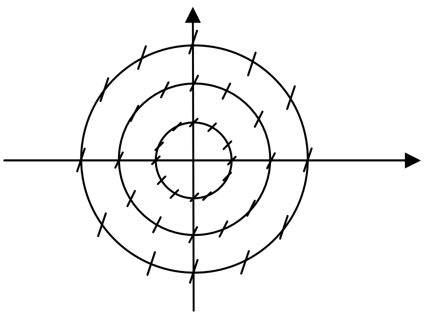
1(i)	$\lambda^2+4\lambda+3=0$	M1	Auxiliary equation	
	$\lambda=-1$ or -3	A1		
	CF $Ae^{-t}+Be^{-3t}$	F1	CF for their roots	
	PI $y=a\cos 2t+b\sin 2t$	B1		
	$\dot{y}=-2a\sin 2t+2b\cos 2t$	M1	Differentiate twice and substitute	
	$\ddot{y}=-4a\cos 2t-4b\sin 2t$			
	$-4a\cos 2t-4b\sin 2t-8a\sin 2t+8b\cos 2t+3a\cos 2t+3b\sin 3t=13\cos 2t$	M1	Compare coefficients	
	$8b-a=13$ $-b-8a=0$	A1		
	$a=-\frac{1}{5},\ b=\frac{8}{5}$	A1		
	GS $y=\frac{1}{5}(8\sin 2t-\cos 2t)+Ae^{-t}+Be^{-3t}$	F1	PI + CF with two arbitrary constants	
				9
(ii)	$t=0,\ y=0\Rightarrow 0=-\frac{1}{5}+A+B$	M1	Use condition	
	$\dot{y}=\frac{1}{5}(16\cos 2t+2\sin 2t)-Ae^{-t}-3Be^{-3t}$	M1	Differentiate	
		F1		
	$t=0,\ \dot{y}=0\Rightarrow 0=\frac{16}{5}-A-3B$	M1	Use condition	
	$\Rightarrow A=-\frac{13}{10},\ \mathrm{B}=\frac{3}{2}$	A1		
	$y=\frac{1}{5}(8\sin 2t-\cos 2t)-\frac{13}{10}e^{-t}+\frac{3}{2}e^{-3t}$	A1	Cao	
				6
(iii)	If $z=y+c$, differentiating (*) gives new DE	M1	Recognise derivative	
	and has 3 arbitrary constants so must be GS	A1		
	or			
	Integrating gives (*) with $+k$ on RHS	*M1*		
	PI will be previous PI $+\frac{1}{3}k$, CF as before, so GS $y+c$	*A1*		
	SC1 for showing that correct *y* from (i) + *c* satisfies new DE			2
(iv)	$z=\frac{1}{5}(8\sin 2t-\cos 2t)+De^{-t}+Ee^{-3t}+c$			
	$t=0,\ z=2\Rightarrow 2=-\frac{1}{5}+D+E+c$	M1	Use condition	
	$\dot{z}=\frac{1}{5}(16\cos 2t+2\sin 2t)-De^{-t}-3Ee^{-3t}$	F1	Derivative	
	$t=0,\ \dot{z}=0\Rightarrow 0=\frac{16}{5}-D-3E$	M1	Use condition	
	$\ddot{z}=\frac{1}{5}(-32\sin 2t+4\cos 2t)+De^{-t}+9Ee^{-3t}$	F1	Second derivative: condone, for this mark only, +*c* appearing	
	$t=0,\ \ddot{z}=13\Rightarrow 13=\frac{4}{5}+D+9E$	M1	Use condition	
	$D=-\frac{13}{10},\ E=\frac{3}{2},\ c=2$	B1		
	$z=\frac{1}{5}(8\sin 2t-\cos 2t)-\frac{13}{10}e^{-t}+\frac{3}{2}e^{-3t}+2$	A1	Cao	
				7

2(a)(i)	$I = \exp(\int -\frac{2}{x}\,dx)$	M1	Attempt integrating factor	
	$= \exp(-2\ln x)$	A1		
	$= x^{-2}$	A1		
	$x^{-2}\frac{dy}{dx} - 2x^{-3}y = x^{-3/2}$	M1	Multiply both sides by IF	
	$\frac{d}{dx}(x^{-2}y) = x^{-3/2}$	M1		
	$x^{-2}y = -2x^{-1/2} + A$	M1	Integrate both sides	
		A1		
	$y = -2x^{3/2} + Ax^2$	F1	Must divide constant	
				8
(ii)	$0 = -2 + A$	M1		
	$y = 2x^2 - 2x^{3/2}$	A1		
				2
(iii)	$x \to 0, y \to 0$	F1		
	$\frac{dy}{dx} = 4x - 3x^{1/2} = 0 \Leftrightarrow x = \frac{9}{16}$ (as $x > 0$)	M1		
	$x \to 0, \frac{dy}{dx} \to 0$	F1		
		B1	Behaviour at origin	
		B1	Through (1,0) and shape for $x > 1$	
		B1	Stationary point at $\left(\frac{9}{16}, -\frac{27}{128}\right)$	
				6
(b)(i)	Circle centre origin	B1		
	Radius 1	B1		
				2
(ii)		B1	One isocline correct	
		B1	All three isoclines correct	
		B1	Reasonably complete and accurate direction indicators	
				3
(iii)		B1	Solution curve	
				1
(iv)		B1	Solution curve	
		B1	Zero gradient at origin	
				2

3(a)(i)	N2L: $ma = -2k^2 x$	M1		
	$2v\dfrac{dv}{dx} = -2k^2 x$	M1	Acceleration $= v\dfrac{dv}{dx}$	
	$v\dfrac{dv}{dx} = -k^2 x$	E1		
				3
(ii)	$\int v dv = \int -k^2 x dx$	M1	Separate and integrate	
	$\frac{1}{2}v^2 = -\frac{1}{2}k^2x^2 + A$	A1	LHS	
		A1	RHS	
	$x = a,\ v = 0 \Rightarrow A = \frac{1}{2}k^2a^2$	M1	Use condition	
	$v^2 = k^2(a^2 - x^2)$	A1		
	So for $v < 0$, $\dfrac{dx}{dt} = -k\sqrt{a^2 - x^2}$	E1		
				6
(iii)	$\int \dfrac{1}{\sqrt{a^2 - x^2}} dx = \int -k dt$	M1	Separate and integrate	
	$\arcsin \frac{x}{a} + B = -kt$	A1	LHS	
		A1	RHS	
	$x = a,\ t = 0 \Rightarrow B = -\frac{1}{2}\pi$	M1	Use condition	
	$x = a\sin(\frac{1}{2}\pi - kt) = a\cos kt$	A1	Either form	
				5
(b)(i)	$\int \omega d\omega = \int -9\sin\theta d\theta$	M1	Separate and integrate	
	$\frac{1}{2}\omega^2 = 9\cos\theta + C$	A1	LHS	
		A1	RHS	
	$\theta = \frac{1}{3}\pi,\ \omega = 0 \Rightarrow C = -\frac{9}{2}$	M1	Use condition	
	So $\omega^2 = 9(2\cos\theta - 1)$	A1		
	$\dfrac{d\theta}{dt} = -3\sqrt{2\cos\theta - 1}$ (decreasing)	E1		
				6
(ii)	$\theta = \frac{1}{3}\pi \Rightarrow \dot{\theta} = 0$	M1		
	So estimate $= \frac{1}{3}\pi + 0 = \frac{1}{3}\pi$	A1		
	The algorithm will keep giving $\theta = \frac{1}{3}\pi$	B1		
	but θ is not constant so not useful	B1		
				4

4(i)	$y = -\frac{1}{2}\dot{x} - \frac{3}{2}x + \frac{3}{2}t$	M1		
	$\dot{y} = -\frac{1}{2}\ddot{x} - \frac{3}{2}\dot{x} + \frac{3}{2}$	M1		
	$-\frac{1}{2}\ddot{x} - \frac{3}{2}\dot{x} + \frac{3}{2} = 2x + (-\frac{1}{2}\dot{x} - \frac{3}{2}x + \frac{3}{2}t) + t + 2$	M1	Eliminate y	
		M1	Eliminate $\dot{y}$	
	$\ddot{x} + 2\dot{x} + x = -5t - 1$	E1		
				5
(ii)	$\lambda^2 + 2\lambda + 1 = 0$	M1	Auxiliary equation	
	$\lambda = -1$ (repeated)	A1	Root	
	CF: $(A + Bt)e^{-t}$	F1	CF for their root(s) (with two constants)	
	PI: $x = at + b$	B1		
	$\dot{x} = a, \ddot{x} = 0$			
	In DE: $0 + 2a + at + b = -5t - 1$	M1	Differentiate and substitute	
	$a = -5$			
	$2a + b = -1$	M1	Compare and solve	
	$a = -5, b = 9$	A1		
	GS: $x = 9 - 5t + (A + Bt)e^{-t}$	F1	GS = PI + CF with two arbitrary constants	
				8
(iii)	$y = -\frac{1}{2}\dot{x} - \frac{3}{2}x + \frac{3}{2}t$	M1		
	$= -\frac{1}{2}[-5 + Be^{-t} - (A + Bt)e^{-t}]$	M1	Differentiate (product rule)	
	$-\frac{3}{2}[9 - 5t + (A + Bt)e^{-t}] + \frac{3}{2}t$	M1	Substitute	
	$= 9t - 11 - (A + \frac{1}{2}B + Bt)e^{-t}$	A1		
				4
(iv)	$t = 0, x = 9 \Rightarrow A = 0$	M1	Use condition	
	$t = 0, y = 0 \Rightarrow 0 = -11 - \frac{1}{2}B \Rightarrow B = -22$	M1	Use condition	
	$x = 9 - 5t - 22te^{-t}$	A1		
	$y = 9t - 11 + (11 + 22t)e^{-t}$	A1		
				4
(v)	$e^{-t} \to 0$	M1		
	$x \approx 9 - 5t$	F1		
	$y \approx 9t - 11$	F1		
				3

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4758/01: Differential Equations (Written Examination)

General Comments

The overall performance on this paper was very good. Many candidates scored high marks and very few scored less than half of the available marks. As usual, the familiar topics tested in Questions 1 and 4 were attempted by almost all of the candidates, with Question 3 the least popular choice. Most candidates have a very good working knowledge of the topics on this syllabus, the exception being, on this occasion, an understanding of the terminology of isoclines and tangent fields.

A high standard of algebraic and arithmetical accuracy of solutions is expected on this paper, and it is pleasing to note an improvement in this aspect.

Comments on Individual Questions

1 This question was attempted by all candidates and many earned the majority of the available marks.

(i) The method was well-understood by all, but a minority of candidates made arithmetical or algebraic errors in solving the linear simultaneous equations in finding the particular integral.

(ii) As in part (i), there were some algebraic errors.

(iii) Most candidates scored one out of the two marks available here, by recognising that one differential equation was the integral/differential of the other. Few candidates were able to go on to give a convincing argument to show that z was equal to $y + c$.

(iv) Apart from arithmetical and algebraic errors, a minority of candidates worked with the particular solution to the original differential equation, rather than the general solution.

2 Candidates usually answered part (a) well, but many seemed unclear of the terminology and/or methods involved in part (b).

(a)(i) Candidates showed a good understanding of the integrating factor method of solving this first order differential equation, and they applied it with accuracy.

(ii) Again, this use of an initial condition was well-executed.

(iii) Most candidates were able to find the stationary point of the curve and the values of y and its derivative as x approaches zero, but they did not always go on to use this information to help them sketch the curve.

(b) There were a few excellent solutions to this part of the question, but the work of many candidates suggested that they were not familiar with the words "isocline" and "tangent field."

(i) A statement that the isocline is a circle with centre at the origin and with unit radius was required here.

(ii) Many candidates showed confusion in their attempted solutions here, indicating that they were unsure of what was required in the requests for three isoclines and corresponding tangent fields. The isoclines, which were concentric circles in this case, were often not drawn.

(iii) The majority of candidates were able to recover here and sketch the solution curve through (0,1).

(iv) Again, candidates recognised the general shape of the solution curve through the origin, but relatively few indicated its gradient at the origin to be zero.

3 This was the least popular choice of question, but those who selected it were usually successful in scoring the majority of the marks.

(a)
(i) This was invariably answered well.

(ii) The separation of variables and integration was done well, but the justification for taking the negative sign in the final expression was not always present.

(iii) Again, the separation of variables and integration was handled correctly by the candidates.

(b)
(i) There were many fully correct solutions here, though a minority of candidates made a sign error in the trigonometric integration. Again, the justification for the negative sign in the given expression was omitted by some candidates.

(ii) Almost all candidates applied the Euler method accurately to obtain the requested estimation. Very few candidates were able to explain that it was not helpful to continue using the given algorithm, because of the non-constancy of θ.

4 The vast majority of candidates attempted this question and many scored full marks.

(i) Solutions were almost always convincing and correct.

(ii) Candidates were clearly very familiar with the method and worked through it accurately.

(iii) Occasionally there were arithmetical errors in finding the general solution for y. It was particularly pleasing that only a handful of candidates failed to use their general solution for x as the starting-point.

(iv) Again, there were a few arithmetical slips, but the method was well-known by all.

(v) This request presented no problems to candidates and with follow through from previous incorrect answers, almost all candidates scored the three marks available.

GCE Mathematics (MEI)									
			Max Mark	a	b	c	d	e	u
4751/01	(C1) MEI Introduction to Advanced Mathematics	Raw	72	55	49	43	37	32	0
		UMS	100	80	70	60	50	40	0
4752/01	(C2) MEI Concepts for Advanced Mathematics	Raw	72	53	46	39	33	27	0
		UMS	100	80	70	60	50	40	0
4753/01	(C3) MEI Methods for Advanced Mathematics with Coursework: Written Paper	Raw	72	54	48	42	36	29	0
4753/02	(C3) MEI Methods for Advanced Mathematics with Coursework: Coursework	Raw	18	15	13	11	9	8	0
4753/82	(C3) MEI Methods for Advanced Mathematics with Coursework: Carried Forward Coursework Mark	Raw	18	15	13	11	9	8	0
4753	(C3) MEI Methods for Advanced Mathematics with Coursework	UMS	100	80	70	60	50	40	0
4754/01	(C4) MEI Applications of Advanced Mathematics	Raw	90	63	56	50	44	38	0
		UMS	100	80	70	60	50	40	0
4755/01	(FP1) MEI Further Concepts for Advanced Mathematics	Raw	72	59	52	45	39	33	0
		UMS	100	80	70	60	50	40	0
4756/01	(FP2) MEI Further Methods for Advanced Mathematics	Raw	72	55	48	41	34	27	0
		UMS	100	80	70	60	50	40	0
4757/01	(FP3) MEI Further Applications of Advanced Mathematics	Raw	72	55	48	42	36	30	0
		UMS	100	80	70	60	50	40	0
4758/01	(DE) MEI Differential Equations with Coursework: Written Paper	Raw	72	63	57	51	45	39	0
4758/02	(DE) MEI Differential Equations with Coursework: Coursework	Raw	18	15	13	11	9	8	0
4758/82	(DE) MEI Differential Equations with Coursework: Carried Forward Coursework Mark	Raw	18	15	13	11	9	8	0
4758	(DE) MEI Differential Equations with Coursework	UMS	100	80	70	60	50	40	0
4761/01	(M1) MEI Mechanics 1	Raw	72	60	52	44	36	28	0
		UMS	100	80	70	60	50	40	0
4762/01	(M2) MEI Mechanics 2	Raw	72	64	57	51	45	39	0
		UMS	100	80	70	60	50	40	0
4763/01	(M3) MEI Mechanics 3	Raw	72	59	51	43	35	27	0
		UMS	100	80	70	60	50	40	0
4764/01	(M4) MEI Mechanics 4	Raw	72	54	47	40	33	26	0
		UMS	100	80	70	60	50	40	0
4766/01	(S1) MEI Statistics 1	Raw	72	53	45	38	31	24	0
		UMS	100	80	70	60	50	40	0
4767/01	(S2) MEI Statistics 2	Raw	72	60	53	46	39	33	0
		UMS	100	80	70	60	50	40	0
4768/01	(S3) MEI Statistics 3	Raw	72	56	49	42	35	28	0
		UMS	100	80	70	60	50	40	0
4769/01	(S4) MEI Statistics 4	Raw	72	56	49	42	35	28	0
		UMS	100	80	70	60	50	40	0
4771/01	(D1) MEI Decision Mathematics 1	Raw	72	51	45	39	33	27	0
		UMS	100	80	70	60	50	40	0
4772/01	(D2) MEI Decision Mathematics 2	Raw	72	58	53	48	43	39	0
		UMS	100	80	70	60	50	40	0
4773/01	(DC) MEI Decision Mathematics Computation	Raw	72	46	40	34	29	24	0
		UMS	100	80	70	60	50	40	0
4776/01	(NM) MEI Numerical Methods with Coursework: Written Paper	Raw	72	62	55	49	43	36	0
4776/02	(NM) MEI Numerical Methods with Coursework: Coursework	Raw	18	14	12	10	8	7	0
4776/82	(NM) MEI Numerical Methods with Coursework: Carried Forward Coursework Mark	Raw	18	14	12	10	8	7	0
4776	(NM) MEI Numerical Methods with Coursework	UMS	100	80	70	60	50	40	0
4777/01	(NC) MEI Numerical Computation	Raw	72	55	47	39	32	25	0
		UMS	100	80	70	60	50	40	0